2.2 DESCRIPTION OF STRESS AT A POINT

Small only consider the continuum equations for an elastic solid. The equation because they depend on the material properties in this chapter we will consider the continuum mechanics. The continuum equations stress and strain are derived constitute.

In principle, the force that causes a structural element to deform the body of knowledge is often called the stress in a body and is applicable to any physical situation where we are interested in the effect of stress on the body. The stress is described in terms of a set of six or more numbers, each of which describes a particular direction and magnitude of force acting on the body. Each of these numbers is known as a component of stress.

In order to determine the magnitude and direction of stress at a point in a body, it is necessary to know the force acting on the body and the size of the body at that point. This information can be obtained from measurements of stress or from calculations based on the equations of elasticity.

In this chapter, we will present the mathematical relationships for the elastic behavior of materials and discuss the stress and strain relationships for two-dimensional and three-dimensional problems.

ELASTIC BEHAVIOR

STRESS AND STRAIN RELATIONSHIPS FOR

TWO

CHAPTER
2.3 STATE OF STRESS IN TWO DIMENSIONS (PLANE STRESS)

The normal stresses and shear stresses in a plane are represented by the components:

\[ \sigma_x, \sigma_y, \tau_{xy} \]

and in the manner

\[ \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \]

The state of stress at a point is completely described by its components.

(1)

The plane strain condition is depicted by a two-dimensional state of stress.

If an axis is normal to the plane of interest, the stress components are:

\[ \sigma_x = \sigma_y = 0 \]

and in the manner

\[ \begin{bmatrix} 0 & \tau_{xy} \\ \tau_{xy} & 0 \end{bmatrix} \]

This is the state of stress at a point.

In the case of anisotropic materials, the stress components are:

\[ \sigma_x, \sigma_y, \tau_{xy} \]

and in the manner:

\[ \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \]

(2)

The plane strain condition is depicted by a two-dimensional state of stress.

If an axis is normal to the plane of interest, the stress components are:

\[ \sigma_x = \sigma_y = 0 \]

and in the manner

\[ \begin{bmatrix} 0 & \tau_{xy} \\ \tau_{xy} & 0 \end{bmatrix} \]

This is the state of stress at a point.

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and in the manner:

\[ \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \]

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and in the manner

\[ \begin{bmatrix} 0 & \tau_{xy} \\ \tau_{xy} & 0 \end{bmatrix} \]

This is the state of stress at a point.

In the case of anisotropic materials, the stress components are:

\[ \sigma_x, \sigma_y, \tau_{xy} \]

and in the manner:

\[ \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \]
For any point in a stress field, it is always possible to define a new coordinate system in which the maximum normal stress is always positive to define a new coordinate system.

Stress equations are written in a form of a tensor, which is then expressed in a matrix form, where the components of the stress tensor are defined in the direction of the normal stress and the shear stress.

The transformation of stress equations now becomes:

\[ \begin{align*}
\sigma_x' & = \sigma_x - \frac{\sigma_y}{2} \\
\sigma_y' & = \sigma_y - \frac{\sigma_x}{2} \\
\tau_{xy}' & = \tau_{xy}
\end{align*} \]

The components of stress in the new coordinate system are:

\[ \begin{align*}
\sigma_x' & = \sigma_x - \frac{\sigma_y}{2} \\
\sigma_y' & = \sigma_y - \frac{\sigma_x}{2} \\
\tau_{xy}' & = \tau_{xy}
\end{align*} \]

The direction of the stress components in the new coordinate system is determined by the following equations:

\[ \begin{align*}
\sigma_x' & = \sigma_x - \frac{\sigma_y}{2} \\
\sigma_y' & = \sigma_y - \frac{\sigma_x}{2} \\
\tau_{xy}' & = \tau_{xy}
\end{align*} \]
expression for tension or compression force. The result of the expression for tension is shown in Fig. (2.8). If the stress is in compression, the stress is in tension. If the stress is in tension, the stress is in compression.

The direction of the principal plane is found by solving for \( \theta \) in Eq. (2.8).

\[
\cos \theta \tan \phi = \frac{\sigma_x - \sigma_y}{2 \tau}
\]

The values of \( \theta \) are substituted into Eq. (2.8) to obtain the values of \( \sigma_x \) and \( \sigma_y \) and the values of \( \tau \). The values of \( \sigma_x \) and \( \sigma_y \) are used to find the values of \( \theta \). If the values of \( \sigma_x \) and \( \sigma_y \) are used to find the values of \( \theta \), the values of \( \tau \) are used to find the values of \( \sigma_x \) and \( \sigma_y \).
Figure 2-12 12.5 MPa - 65 kPa

\[ \sigma = \frac{65}{2.5} = 26 \text{ MPa} \]

We can also find \( \alpha \) as:

\[ \frac{\sigma}{2.5} = \frac{26}{2.5} = 10.4 \text{ MPa} \]

\[ \theta = \tan^{-1} \left( \frac{2}{2} \right) = 45^\circ \]

\[ \varphi = \tan^{-1} \left( \frac{1}{1} \right) = 45^\circ \]

\( \theta = \frac{\pi}{4} \text{ rad} \) and \( \varphi = \frac{\pi}{4} \text{ rad} \)

From Eqs. (2.12) and (2.13), the angle of stress is given by:

\[ \theta = \frac{\pi}{4} \text{ rad} \] and \( \varphi = \frac{\pi}{4} \text{ rad} \)

Equation (2.12) is the equation of a circle of the form \( x^2 + y^2 = r^2 \).

We can solve for \( \sigma \) in terms of these equations and

\[ \frac{\sigma}{2.5} = \frac{26}{2.5} = 10.4 \text{ MPa} \]

\[ \theta = \tan^{-1} \left( \frac{2}{2} \right) = 45^\circ \]

\[ \varphi = \tan^{-1} \left( \frac{1}{1} \right) = 45^\circ \]

\[ \theta = \frac{\pi}{4} \text{ rad} \] and \( \varphi = \frac{\pi}{4} \text{ rad} \)

The angle of stress is given by:

\[ \theta = \frac{\pi}{4} \text{ rad} \] and \( \varphi = \frac{\pi}{4} \text{ rad} \)

Example: The angle of stress is given by:

\[ \theta = \frac{\pi}{4} \text{ rad} \] and \( \varphi = \frac{\pi}{4} \text{ rad} \)

Equation (2.11) gives the angle of the maximum shear stress for a plane on which \( \sigma \) acts.

\[ \theta = \frac{\pi}{4} \text{ rad} \] and \( \varphi = \frac{\pi}{4} \text{ rad} \)

The means that \( \theta = 45^\circ \) and \( \varphi = 45^\circ \) are orthogonal and have \( \sigma \) and \( \varphi \) separated in
Any orientation of force in a coordinate system is equal to the sum of the normal stress or the shear stresses in that system. For a two-dimensional state of stress, if \( \sigma_x, \tau_{xy}, \tau_{yx}, \sigma_y \) are the normal stresses and shear stresses, respectively, then the normal stress in any direction \( \theta \) is given by:

\[
\sigma_{\theta} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \cos \theta \sin \theta.
\]

The shear stress in any direction is:

\[
\tau_{\theta} = -\tau_{xy} \cos \theta \sin \theta + \tau_{yx} \cos \theta \sin \theta.
\]

If the orientation of the force is such that \( \theta = 45^\circ \), then:

\[
\sigma_{45^\circ} = \frac{\sigma_x + \sigma_y}{2} + \frac{\tau_{xy}}{\sqrt{2}},
\]

\[
\tau_{45^\circ} = \frac{\tau_{xy}}{\sqrt{2}}.
\]

In the context of the coordinate axes, let \( x \) and \( y \) be the principal axes where the normal stress is maximum and minimum, respectively. Then:

\[
\sigma_x \geq \sigma_y, \quad \tau_{xy} = 0.
\]

The principal stresses can be determined by solving the characteristic equation:

\[
0 = \left( \sigma_x - \sigma \right) \left( \sigma_y - \sigma \right) - \tau_{xy}^2.
\]

The roots of this equation are the principal stresses, \( \sigma_1 \) and \( \sigma_2 \).

If \( \sigma_1 > \sigma_2 \), then:

\[
\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2},
\]

\[
\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}.
\]

The selection of the coordinate axes is important for determining the principal stresses. The stress component in the positive direction of the coordinate axis is positive, and the shear stress is zero.

5. STATE OF STRESS IN THREE DIMENSIONS

In a three-dimensional state of stress, the stress components are: \( \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz} \). The equations for the stress components in the principal axes are more complex but follow similar principles. The principal stresses are determined by solving the characteristic equation:

\[
0 = \left( \sigma_x - \sigma \right) \left( \sigma_y - \sigma \right) \left( \sigma_z - \sigma \right) - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2.
\]

The roots of this equation are the principal stresses, \( \sigma_1, \sigma_2, \sigma_3 \).
\[ \begin{align*}
\tau_{ij} &= \frac{1}{2} (F_{ij} - F_{ji}) \\
\sigma_{ij} &= \frac{1}{2} (F_{ij} + F_{ji})
\end{align*} \]

The stress tensor is a second-rank tensor that describes the stress at a point in a solid. It is defined as the limit of the average normal stress acting on an infinitesimal surface element. The stress tensor has six independent components in three dimensions, which are usually denoted as \(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}\). These components represent the normal stress and shear stress acting on each plane in the solid. The stress tensor is symmetric, meaning \(\sigma_{ij} = \sigma_{ji}\), and it is used to describe the distribution of forces within a solid medium.

In this section, we will discuss the transformation of stress under a change of coordinate system. The stress tensor is invariant under rotations, meaning that its components remain the same when the coordinate system is rotated. This property is crucial in the study of stress and strain in solids, as it allows us to express the stress state in any coordinate system without losing any information.

**Stress Tensor**

- \(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}\) represent the normal stress acting on planes parallel to the coordinate axes.
- \(\sigma_{xy}, \sigma_{yz}, \sigma_{zx}\) represent the shear stress acting on planes at 45 degrees to the coordinate axes.

**Mechanical Fundamentals**

Stress and strain relationships for elastic behavior are crucial in the study of solid mechanics. The stress tensor relates the internal forces acting on a small element of a solid to the displacement of that element. Understanding these relationships is essential for predicting the behavior of materials under various loading conditions.
The transformation of the stress tensor is given by the equation:

\[
\sigma'_{ij} = \sum_k \frac{\partial \sigma_{ik}}{\partial x_j} \frac{x_k}{x_j}
\]

where \(\sigma'_{ij}\) is the transformed stress tensor, \(\sigma_{ik}\) is the original stress tensor, and \(x_k\) and \(x_j\) are the original and transformed coordinates, respectively.

This equation is derived from the concept of stress transformation, which is used to determine the stress distribution in a material undergoing deformation. The transformation takes into account the change in the orientation of the material due to the deformation, allowing for a more accurate prediction of how stresses are distributed within the material.

The stress tensor is a second-rank tensor, which means it has components that describe the stress in different directions. These components are 

\[
\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}
\]

where \(\sigma_{xx}\) is the stress in the x-direction, \(\sigma_{yy}\) is the stress in the y-direction, and so on. The transformation of these components into a new coordinate system is given by the above equation.

The transformation matrix \(\frac{\partial x}{\partial x'} = \frac{\partial y}{\partial x'} = \frac{\partial z}{\partial x'} = \frac{\partial x}{\partial y'} = \frac{\partial y}{\partial y'} = \frac{\partial z}{\partial y'} = \frac{\partial x}{\partial z'} = \frac{\partial y}{\partial z'} = \frac{\partial z}{\partial z'}\)

is used to transform the stress tensor from the original coordinate system to the new coordinate system. This matrix is derived from the first-order partial derivatives of the transformation equations.
2.7 Moir's Circle—Three Dimensions

where

The first principal stress is the largest, and the second principal stress is the smallest. The direction of the principal stresses is found by solving the characteristic equation, which is given by:

\[ \lambda^2 - (\sigma_1 + \sigma_2 + \sigma_3) \lambda + (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 - \sigma_1 \sigma_2 - \sigma_2 \sigma_3 - \sigma_3 \sigma_1) = 0 \]

The roots of this equation are \( \lambda_1 \) and \( \lambda_2 \), which are the principal stresses.

The orientation of the principal stresses can be determined using the Mohr's circle. The circle is centered at the origin and has a radius equal to half the difference between the maximum and minimum principal stresses. The normal to the circle at any point represents the direction of the principal stress at that point.

The principal stresses are also related to the maximum and minimum shear stresses by the following equations:

- Maximum shear stress: \( \sigma_{ss} = \frac{1}{2} (\sigma_1 - \sigma_2) \)
- Minimum shear stress: \( \sigma_{mm} = \frac{1}{2} (\sigma_2 - \sigma_1) \)

These equations are derived from the stress transformation equations and are useful in the analysis of stress distribution in materials.

The concept of principal stresses is crucial in the study of stress and strain in solid mechanics. They provide a simplified representation of the stress field by reducing the three-dimensional stress state to two mutually perpendicular planes. This simplification is particularly useful in the design and analysis of structures, as it allows engineers to focus on the most critical stress components without the complexity of the full three-dimensional stress field.
2.9 DESCRIPTION OF STRAIN AT A POINT

The deformation of a solid may be made up of

Axial deformation, shear deformation, and dilatation. The deformation of a continuum may result from rigid-body translation.

Figure 2.11: Schematic of strain at a point

- Axial deformation
- Shear deformation
- Dilatation

The deformation parameters are:

\[ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \]

where:

- \( \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz} \) are the strains in the axial directions
- \( \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \) are the shear strains

These strains are defined in terms of the change in length and angle of the infinitesimal deformation element, respectively, relative to their original state.

The strain tensor is symmetric, and it can be represented by a 9x9 matrix.

\[ \epsilon = \frac{1}{2} \left[ \begin{array}{ccc} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{array} \right] \]

The strain energy density function, \( W \), is defined as

\[ W = \frac{1}{2} \epsilon : \sigma \]

where \( \sigma \) is the stress tensor.

The first Piola-Kirchhoff stress tensor, \( \Pi \), is

\[ \Pi = \frac{E}{1-\nu} \left[ \begin{array}{ccc} 1+\nu & \nu & \nu \\ \nu & 1+\nu & \nu \\ \nu & \nu & 1+\nu \end{array} \right] \epsilon \]

and the Cauchy stress tensor, \( \sigma \), is

\[ \sigma = \frac{E}{1-\nu} \left[ \begin{array}{ccc} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-\nu \end{array} \right] \Pi \]

These formulations are fundamental in elasticity theory and are used to analyze the behavior of materials under various loading conditions.
For this one-dimensional case, the displacement is given by:

\[ u = \frac{\partial u}{\partial x} \]

Similarly, for the angular distortion of the x-axis,

\[ \delta \theta = \frac{\partial \theta}{\partial x} \]

The shear strain displacement and the shear displacement are given by:

\[ \gamma = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \]

Consider an element in the x-y plane, the shear strain is given by:

\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]

The area strain is given by:

\[ \varepsilon = \varepsilon_x + \varepsilon_y + 2\nu \varepsilon_{xy} \]

where \( \nu \) is the Poisson's ratio.

The change in volume strain is given by:

\[ \varepsilon_v = \frac{1}{3} \left( \varepsilon_x + \varepsilon_y + \varepsilon_z \right) \]

These equations are known as the compatibility equations for elastoplastic materials.
Stress and Strain Relationships for Plastic Behavior

\[ \sigma = \frac{6E}{(1 + 2v)(1 - v)} \epsilon \]

The stress-strain relationship is given by the equation above, where \( \sigma \) is the stress, \( \epsilon \) is the strain, \( E \) is Young's modulus, and \( v \) is Poisson's ratio.

**Compressive Stress:**

\[ \sigma = \frac{6E}{(1 + 2v)(1 - v)} \epsilon \]

**Tensile Stress:**

\[ \sigma = \frac{6E}{(1 + 2v)(1 - v)} \epsilon \]

**Shear Stress:**

\[ \tau = \frac{3G}{2(1 + v)} \gamma \]

where \( G \) is the shear modulus and \( \gamma \) is the shear strain.

**Strain Energy Density:**

\[ W = \frac{1}{2} \sigma \epsilon - \frac{1}{3} G \phi \]

where \( W \) is the strain energy density, \( \sigma \) is the stress, \( \epsilon \) is the strain, and \( G \) is the shear modulus.

**Failure Criteria:**

- **Mises' Tresca's criterion:**

\[ \sqrt{3} \sigma \leq f \]

where \( f \) is the yield stress.

- **von Mises' criterion:**

\[ \sqrt{\sigma^2 - 3 \tau^2} \leq f \]

where \( \sigma \) is the maximum principal stress and \( \tau \) is the maximum shear stress.

**Mohr's Circle:**

\[ \sigma = \frac{1}{2} f \pm \sqrt{\left( \frac{1}{2} f \right)^2 - G \phi} \]

where \( f \) is the yield stress and \( G \) is the shear modulus.

**Mohr's Stress Tensor:**

\[ \begin{bmatrix} \sigma & \tau \sin \phi \cos \theta & \tau \sin \phi \sin \theta \\ \tau \cos \phi \cos \theta & \sigma & \tau \cos \phi \sin \theta \\ \tau \sin \phi \sin \theta & \tau \cos \phi \sin \theta & \sigma \end{bmatrix} \]

where \( \sigma \) is the normal stress, \( \tau \) is the shear stress, and \( \phi \) and \( \theta \) are the angles of the shear stress vector.

**Mohr's Strain Tensor:**

\[ \begin{bmatrix} \epsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_z \end{bmatrix} \]

where \( \epsilon \) is the strain and \( \gamma \) is the shear strain.

**Equations of State:**

- **Isotropic:**

\[ \sigma = K \epsilon + \nu P \]

where \( K \) is the bulk modulus, \( \nu \) is Poisson's ratio, and \( P \) is the pressure.

- **Anisotropic:**

\[ \sigma = \mathbf{C} : \epsilon \]

where \( \mathbf{C} \) is the elastic compliance tensor and \( \epsilon \) is the strain.

**Deformation Gradient Tensor:**

\[ \mathbf{F} = \mathbf{I} + \mathbf{E} \]

where \( \mathbf{F} \) is the deformation gradient tensor and \( \mathbf{E} \) is the strain tensor.

**Right Multiplication:**

\[ \mathbf{F} \mathbf{F}^T = \mathbf{I} + \mathbf{E} \mathbf{E}^T \]

**Left Multiplication:**

\[ \mathbf{F}^T \mathbf{F} = \mathbf{I} + \mathbf{E} \mathbf{E}^T \]

**Stretches:**

- **principal stretches:**

\[ \lambda_1, \lambda_2, \lambda_3 \]

- **stretch ratios:**

\[ a = \frac{\lambda_1}{\lambda_3} \]

**Strain Energy:**

\[ W = \frac{1}{2} \mathbf{E} : \mathbf{E} \]

where \( \mathbf{E} \) is the strain tensor.

**Rotation Tensor:**

\[ \mathbf{R} = \mathbf{F}^{-1} \mathbf{F}^T \]

where \( \mathbf{R} \) is the rotation tensor.

**Cosine Matrix:**

\[ \mathbf{C} = \frac{1}{2} \left( \mathbf{I} + \mathbf{R} \right) \]

where \( \mathbf{I} \) is the identity tensor.

**Shear Strain:**

\[ \gamma = \frac{1}{2} \left( \mathbf{E} - \mathbf{E}^T \right) \]

where \( \mathbf{E} \) is the strain tensor.

**Volume Change:**

\[ \Delta V = \frac{1}{2} \mathbf{E} : \mathbf{E} \]

**Elastic Response:**

- **Linear Elasticity:**

\[ \mathbf{F} = \mathbf{I} + \mathbf{E} \]

- **Nonlinear Elasticity:**

\[ \mathbf{F} = \mathbf{I} + \mathbf{E} + \mathbf{E} \mathbf{F} - \mathbf{F} \mathbf{E} \]

**Destabilizing Stability:**

- **Euler-Bernoulli Beam:**

\[ \frac{d^2 y}{dx^2} = 0 \]

**Euler-Bernoulli Column:**

\[ \frac{d^4 y}{dx^4} = 0 \]

**Euler-Lagrange Equation:**

\[ \frac{d}{dx} \left( \frac{d^2 y}{dx^2} \right) - \frac{d^2 y}{dx^2} = 0 \]

**Stability Criteria:**

- **Crittenden Criterion:**

\[ \frac{d^2 y}{dx^2} = 0 \]

- **Krylov Criterion:**

\[ \frac{d^4 y}{dx^4} = 0 \]

**Elastic Stability:**

- **Euler-Bernoulli:**

\[ \frac{d^2 y}{dx^2} = 0 \]

- **Euler-Lagrange:**

\[ \frac{d^4 y}{dx^4} = 0 \]

**Inelastic Stability:**

- **Krieger's Criterion:**

\[ \frac{d^3 y}{dx^3} = 0 \]

**Krieger's Equation:**

\[ \frac{d^3 y}{dx^3} = 0 \]

**Krieger's Inelastic Stability:**

\[ \frac{d^3 y}{dx^3} = 0 \]

**Elastic Flexural Stability:**

- **Euler-Bernoulli:**

\[ \frac{d^2 y}{dx^2} = 0 \]

- **Euler-Lagrange:**

\[ \frac{d^4 y}{dx^4} = 0 \]

**Elastic Flexural Inelasticity:**

- **Krieger's Criterion:**

\[ \frac{d^3 y}{dx^3} = 0 \]

**Elastic Flexural Stability:**

- **Euler-Bernoulli:**

\[ \frac{d^2 y}{dx^2} = 0 \]

- **Euler-Lagrange:**

\[ \frac{d^4 y}{dx^4} = 0 \]
9.4.6 mean strain of the longitudinal (axial) component of strain

\[
\bar{\varepsilon}_x = \frac{\varepsilon_x}{\varepsilon_x + \varepsilon_y + \varepsilon_z}
\]

with which small strains are the principal strains of strains become

\[
I_p = \frac{1}{(e + 1)(f + 1)(g + 1)} - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)}
\]

change the volume strain \(\Delta V\) due to shear

\[
\Delta V = \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right)
\]

principal shear strains can be obtained from (2.75)

\[
\xi = \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right)
\]

Compress the second invariant of strain and equation of the volume strain

\[
\sigma = (e - f) \mu, \quad \lambda = \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right)
\]

and simplify the principal shear strains are obtained from the initial equations

\[
\begin{align*}
\xi &= \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right) \\
\eta &= \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right) \\
\zeta &= \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right)
\end{align*}
\]

The direction of the principal shear strains is obtained from the initial equations

\[
\begin{align*}
\xi &= \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right) \\
\eta &= \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right) \\
\zeta &= \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right)
\end{align*}
\]

principal shear strains of the principal strain

\[
\begin{align*}
0 &= (e - f) \mu, \quad \lambda = \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right) \\
0 &= (e - f) \mu, \quad \lambda = \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right) \\
0 &= (e - f) \mu, \quad \lambda = \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right)
\end{align*}
\]

principal shear strains and equation of volume strain

\[
\begin{align*}
\xi &= \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right) \\
\eta &= \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right) \\
\zeta &= \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \left( - \frac{1}{e f p x - p f p x (e + 1)(f + 1)(g + 1)} \right)
\end{align*}
\]
2.10 HYDRAULIC AND DEVIATOR COMPONENTS OF STRESS

Consider the angle $\alpha$ on the Mohr's circle. The major principal stress $\sigma_1$ is the line through $O$ and $A$ where the Mohr's circle intersects. The minor principal stress $\sigma_2$ is the line through $O$ and $B$ where the Mohr's circle intersects. $\alpha$ is the angle between $\sigma_1$ and $\sigma_2$.

Definitions of principal stresses and $\alpha$.

$\alpha$ is the angle between the major principal stress $\sigma_1$ and the minor principal stress $\sigma_2$.

Figure 2.16 (left) shows the Mohr's circle for the stress state.

$\sigma_1$ and $\sigma_2$ are the maximum and minimum principal stresses, respectively.

In practical problems, experimental stress analysis is often important.

For practical problems of experimental stress analysis, it is important to determine the angle $\alpha$ in a Mohr's circle.

Figure 2.17 (left) shows a Mohr's circle for the stress state.

$\sigma_1$ and $\sigma_2$ are the maximum and minimum principal stresses, respectively.

In practical problems, experimental stress analysis is often important to determine the angle $\alpha$ in a Mohr's circle.
2.11 STRESS-STRAIN RELATIONSHIP

The third invariant \( I \) is the determinant of Eq. (2.16):

\[
\begin{vmatrix}
\sigma_x + \sigma_y + \sigma_z & \sigma_x & \sigma_x \\
\sigma_x & \sigma_y + \sigma_z & \sigma_y \\
\sigma_x & \sigma_y & \sigma_z
\end{vmatrix} = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - \sigma_z \sigma_x
\]

\( I = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - \sigma_z \sigma_x = \gamma \)

The principal strains \( \varepsilon_i \) are the roots of the equation:

\[
\varepsilon_i^3 - 3\gamma \varepsilon_i + \beta = 0
\]

where \( \gamma \) and \( \beta \) are properties of the material.

(2.18) \[
\begin{cases}
\varepsilon_i &= \frac{1}{2} \left( \frac{\varepsilon_x}{(\varepsilon_x - \varepsilon_y + \varepsilon_z)} \right) \\
\varepsilon_i &= \frac{1}{2} \left( \frac{\varepsilon_y}{(\varepsilon_x - \varepsilon_y + \varepsilon_z)} \right) \\
\varepsilon_i &= \frac{1}{2} \left( \frac{\varepsilon_z}{(\varepsilon_x - \varepsilon_y + \varepsilon_z)} \right)
\end{cases}
\]

(2.19) \[
\begin{cases}
\sigma_x &= \frac{E}{(1 - \nu) \sigma_x} + \frac{E}{(1 - \nu) \sigma_y} + \frac{E}{(1 - \nu) \sigma_z}
\end{cases}
\]

2.2 DEFORMATION UNDER APPLYING STRESS

The deformation of the stress tensor is given by:

\[
\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_z - \varepsilon_z \varepsilon_x = \gamma
\]

The deformation of the stress tensor is given by:

\[
\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_z - \varepsilon_z \varepsilon_x = \gamma
\]

The deformation of the stress tensor is given by:

\[
\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_z - \varepsilon_z \varepsilon_x = \gamma
\]

The deformation of the stress tensor is given by:

\[
\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_z - \varepsilon_z \varepsilon_x = \gamma
\]

The deformation of the stress tensor is given by:

\[
\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_z - \varepsilon_z \varepsilon_x = \gamma
\]

The deformation of the stress tensor is given by:

\[
\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_z - \varepsilon_z \varepsilon_x = \gamma
\]

The deformation of the stress tensor is given by:

\[
\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_z - \varepsilon_z \varepsilon_x = \gamma
\]

The deformation of the stress tensor is given by:

\[
\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_z - \varepsilon_z \varepsilon_x = \gamma
\]

The deformation of the stress tensor is given by:

\[
\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_z - \varepsilon_z \varepsilon_x = \gamma
\]

The deformation of the stress tensor is given by:

\[
\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_z - \varepsilon_z \varepsilon_x = \gamma
\]
The order of application of these two stress relations may be combined by direct substitution.

\[ \frac{d}{d} \left( \frac{d}{d} - d \right) = \frac{\partial}{\partial} (\partial + \partial) \frac{d}{d} - \frac{\partial}{\partial} (\partial + \partial) \frac{d}{d} = \frac{\partial}{\partial} (\partial + \partial) \frac{d}{d} = \frac{\partial}{\partial} (\partial + \partial) \]

Equations (2.61) and (2.65) may be expressed as

\[ \frac{d}{d} - \frac{\partial}{\partial} (\partial + \partial) \frac{d}{d} = \frac{\partial}{\partial} (\partial + \partial) \frac{d}{d} = \frac{\partial}{\partial} (\partial + \partial) \]

Many other relationships can be developed between these four isotropic elastic constants. For example, another important relationship is the expression relating \( \frac{d}{d} \) and \( \frac{\partial}{\partial} \).

\[ \frac{d}{d} \left( \frac{d}{d} - \frac{\partial}{\partial} (\partial + \partial) \right) \]

(2.62)

The elements of the left are the volume strain, \( d \), and the term on the right is \( d \).

### TABLE 2-1: TYPICAL ROOM-TEMPERATURE VALUES OF ELASTIC CONSTANTS FOR SOME MATERIALS

<table>
<thead>
<tr>
<th>Material</th>
<th>( E ) (GPa)</th>
<th>( v )</th>
<th>( \nu )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stainless steel (304)</td>
<td>210</td>
<td>0.30</td>
<td>0.30</td>
<td></td>
</tr>
<tr>
<td>Copper</td>
<td>110</td>
<td>0.35</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>Aluminum</td>
<td>70.7</td>
<td>0.35</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>Graphite</td>
<td>200</td>
<td>0.20</td>
<td>0.20</td>
<td></td>
</tr>
</tbody>
</table>

The shear strain \( \gamma \) is given by the expression

\[ \gamma = \frac{d}{d} (\partial + \partial) \frac{d}{d} - \frac{\partial}{\partial} (\partial + \partial) \frac{d}{d} = \frac{\partial}{\partial} (\partial + \partial) \frac{d}{d} \]

(2.63)

### TABLE 2-2: DIRECTIONAL PROPERTIES OF ELASTIC CONSTANTS

<table>
<thead>
<tr>
<th>Plane</th>
<th>( E_{xx} ) (GPa)</th>
<th>( E_{yy} ) (GPa)</th>
<th>( E_{xy} ) (GPa)</th>
<th>( G_{xy} ) (GPa)</th>
<th>( G_{yz} ) (GPa)</th>
<th>( G_{zx} ) (GPa)</th>
<th>( v_{xy} )</th>
<th>( v_{yz} )</th>
<th>( v_{zx} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>XY</td>
<td>100</td>
<td>200</td>
<td>60</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>XZ</td>
<td>100</td>
<td>100</td>
<td>60</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>YZ</td>
<td>100</td>
<td>100</td>
<td>60</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
</tr>
</tbody>
</table>

Thus, to produce a normal strain \( \epsilon \) and two transverse strain \( \epsilon_x \) and \( \epsilon_y \), stress \( \sigma_x \) and \( \sigma_y \) are produced by applying the principle of superposition to determine the stress distribution in the material.
\[
\begin{align*}
\sigma_0 &= \frac{1}{2}(\sigma + \tau) = \frac{3\tau_0}{2} \\
\tau_0 &= \frac{1}{2}(\sigma - \tau) = \frac{3}{2} \tau_0 \\
\end{align*}
\]

where \( \sigma \) and \( \tau \) are the principal stresses, and \( \sigma_0 \) and \( \tau_0 \) are the principal strains. The principal strains are obtained from the plane stress equation:

\[
\begin{align*}
\varepsilon_x &= \frac{1}{2}(\sigma_x - \tau_x) = \frac{3\tau_y}{2} \\
\varepsilon_y &= \frac{1}{2}(\sigma_y - \tau_y) = \frac{3\tau_x}{2} \\
\end{align*}
\]

and the normal stress is:

\[
\sigma = \frac{1}{2}(\sigma_x + \sigma_y) = \frac{3}{2} \tau_0
\]

For the case of plane stress, the two simple and useful equations relating stress and strain are:

\[
\begin{align*}
\sigma &= \frac{3\tau}{2} \\
\tau &= \frac{3}{2} \sigma
\end{align*}
\]

where \( \sigma \) and \( \tau \) are the normal stress and shear stress, respectively. The strain relationship between the normal stress and normal strain is:

\[
\varepsilon = \frac{1}{2} \sigma = \frac{3}{2} \tau
\]

2.2 CALCULATION OF STRESSES FROM ELASTIC STRAINS

\[
\begin{align*}
\sigma_x &= \frac{3\tau_y}{2} \\
\tau_y &= \frac{3\tau_x}{2}
\end{align*}
\]

For the case of plane strain, the stress and strain equations are:

\[
\begin{align*}
\sigma_x &= \frac{3\tau_y}{2} \\
\tau_y &= \frac{3\tau_x}{2}
\end{align*}
\]

where \( \tau_0 \) is the maximum shear stress, and \( \sigma_0 \) is the normal stress. The equations for calculating stresses from strains are:

\[
\begin{align*}
\sigma_x &= \frac{3\tau_y}{2} \\
\tau_y &= \frac{3\tau_x}{2}
\end{align*}
\]

For example, if \( \tau_0 = 1 \) and \( \sigma_0 = 1 \), then:

\[
\begin{align*}
\tau_y &= \frac{3}{2} \\
\sigma_x &= \frac{3}{4}
\end{align*}
\]

where \( \tau_0 = 1 \) and \( \sigma_0 = 1 \).
2-14 AN INTRODUCTION TO ELASTIC BEHAVIOR

When a force is applied to a material, it deforms or changes shape. The deformation, or strain, results from the applied force. The strain is expressed as a ratio of the change in length or area to the original length or area. The stress is the force per unit area applied to the material.

\[ \sigma = \frac{F}{A} \]

where \( \sigma \) is the stress, \( F \) is the force, and \( A \) is the area.

The stress-strain relationship describes how much a material will deform in response to a given stress. For many materials, this relationship is linear and can be described by Hooke's Law:

\[ \sigma = E \varepsilon \]

where \( \varepsilon \) is the strain and \( E \) is Young's modulus, a measure of the material's stiffness.

The strain energy is the energy stored in a material due to deformation. It is given by

\[ U = \frac{1}{2} \int \sigma \varepsilon \, dV \]

where \( U \) is the strain energy, \( \sigma \) is the stress, \( \varepsilon \) is the strain, and \( dV \) is the volume element.

The total deformation energy is the sum of the strain energy due to all sources of deformation, including external forces, boundary conditions, and internal stresses.

\[ U_{total} = U_{external} + U_{boundary} + U_{internal} \]

The deformation energy is stored in the material and can be released when the material returns to its original shape. This energy is important in understanding the behavior of materials under load and in designing structures and machines.
stress and strain relations for elastic behavior
<table>
<thead>
<tr>
<th>Number of Independent Strains</th>
<th>Non-Metallic</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

The elastic stiffness constants are defined by equations like:

\[ C_{ij} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{pmatrix} 1 & \nu & \nu \\ \nu & 1 & \nu \\ \nu & \nu & 1 \end{pmatrix} \]

For example, for a material with Young's modulus \( E \) and Poisson's ratio \( \nu \), the elastic stiffness constant \( C_{ij} \) can be calculated.

\[ C_{11} = \frac{E}{(1 + \nu)(1 - 2\nu)} \]

\[ C_{12} = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \]

\[ C_{13} = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \]

\[ C_{22} = \frac{E}{(1 + \nu)(1 - 2\nu)} \]

\[ C_{23} = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \]

\[ C_{33} = \frac{E}{(1 + \nu)(1 - 2\nu)} \]

\[ C_{44} = \frac{E}{2(1 + \nu)} \]

\[ C_{55} = \frac{E}{2(1 + \nu)} \]

\[ C_{66} = \frac{E}{2(1 + \nu)} \]

The useful combination for describing the strains of elastic components and

\[ \text{Stress and Strain Relationships for Elastic Behavior} \]
Example 2.1: The modulus of elasticity, $E$, can be obtained from the following equations:

\[
E = \frac{1}{\frac{1}{E_0} + \frac{1}{E_1}}
\]

where $E_0$ and $E_1$ are the initial and final moduli of elasticity, respectively.

2.2: The stress-strain relationship for a material can be expressed as:

\[
\sigma = E\varepsilon
\]

where $\sigma$ is the stress and $\varepsilon$ is the strain.

2.3: The Poisson's ratio, $\nu$, is defined as:

\[
\nu = \frac{\varepsilon_y}{\varepsilon_x} = \frac{1}{2}\frac{E}{G}
\]

where $E$ is Young's modulus and $G$ is the shear modulus.

For a composite material, the effective properties can be determined by:

\[
\frac{1}{E_c} = \frac{1}{E_1} + \frac{1}{E_2}
\]

where $E_1$ and $E_2$ are the moduli of the two constituent materials.

Table 2.3: Stress and strain relationships for composite materials.
\[ \tan \left( \frac{\sigma_1 - \sigma_2}{\sigma_0} \right) = \frac{\sigma_1 - \sigma_2}{2\sigma_0} \]

\[ \cos \left( \frac{\sigma_1 + \sigma_2}{\sqrt{\sigma_0^2 + \sigma_1^2}} \right) = \frac{\sigma_1 + \sigma_2}{2\sigma_0} \]

In addition to producing a stress concentration, a notch also creates a localized condition of residual or residual stress. For example, for the circular hole in a plate subjected to an axial load, a residual stress field is induced in the vicinity of the notch. This stress field is characterized by a combination of tension around the hole and compression away from it. The stress concentration factor, \( K_c \), is defined as the ratio of the maximum stress to the applied stress at a point in the vicinity of the notch.

The presence of stress concentrations can lead to higher stresses at critical locations, potentially causing failure under lower applied loads. Thus, understanding stress concentrations is crucial in the design and analysis of structures and materials.

2.15 STRESS CONCENTRATION

There are several methods to determine the stress concentration factor, \( K_c \), for a given geometry.

For Notch:

\[ K_c = \frac{\sigma_{max}}{\sigma_{applied}} \]

The direction cosines for the chief directions in a cubic lattice are:

\[
\begin{array}{cccc}
\xi/1 & \xi/1 & \xi/1 & (111) \\
0 & 0 & 1 & (110) \\
0 & 1 & 0 & (100) \\
1 & 0 & 0 & (011) \\
\end{array}
\]
In this study, we are interested in the behavior of ductile materials under stress. The yield criterion is a fundamental concept in plasticity theory, which describes the onset of plastic deformation. The von Mises yield criterion is one of the most widely used in engineering practice due to its simplicity and the fact that it is effective for isotropic materials.

The von Mises yield criterion is expressed as:

\[
\sigma_y = \sqrt{\frac{3}{2} \left(\varepsilon_{11}^p - \varepsilon_0^p\right) \left(\varepsilon_{22}^p - \varepsilon_0^p\right) + \left(\varepsilon_{33}^p - \varepsilon_0^p\right)^2}
\]

Where \(\varepsilon_{ij}^p\) are the plastic strains and \(\varepsilon_0^p\) is the yield strain. This criterion assumes that the von Mises stress is a sufficient condition for yielding.

To summarize, the yield criterion provides a mathematical framework for predicting the onset of plastic deformation in ductile materials, which is crucial for the design and analysis of structures. The next steps will involve more detailed analysis and application of these principles to specific engineering problems.
The shear stress-strain relationship is that for a given shear strain the shear stress is constant. The shear stress-strain relationship is given by

\[ \tau = \frac{\gamma}{\phi} \]  

where \( \tau \) is the shear stress and \( \gamma \) is the shear strain. The shear stress-strain relationship is used to determine the stress-strain behavior of materials.

Another important property of the stress-strain relationship is the yield stress. The yield stress is the stress at which the material begins to deform plastically. The yield stress is a critical parameter in the design of structures and components. The yield stress is determined experimentally or through theoretical calculations.

The yield stress can be calculated using the following equation:

\[ \sigma_y = \frac{3}{2} \frac{E}{(1-\nu)^2} \]  

where \( \sigma_y \) is the yield stress, \( E \) is the Young's modulus, and \( \nu \) is the Poisson's ratio.

This equation is used to determine the yield stress for materials under different loading conditions. The yield stress is an important parameter in the design of structures and components.

The stress-strain relationship can be expressed in terms of the stress ratio:

\[ \frac{\sigma}{\sigma_y} = \frac{\varepsilon}{\varepsilon_y} \]  

where \( \sigma \) is the stress, \( \sigma_y \) is the yield stress, \( \varepsilon \) is the strain, and \( \varepsilon_y \) is the yield strain.

The stress-strain relationship is used to determine the stress distribution in structures and components. The stress-strain relationship is an important parameter in the design of structures and components.
6. THE YIELD LOCUS

6.6. COMBINED STRESS TESTS

The condition for yielding under states of stress other than uniaxial and constant is

\[ \sigma_1 + \tau_{12} = \sigma_{y} \]

where \( \sigma_1 \) is the major principal stress.

Example: Use the maximum-stress criterion to calculate whether yield will occur for the following stress state:

\[ \sigma_1 = 100 \text{ MPa}, \quad \sigma_2 = 50 \text{ MPa}, \quad \tau_{12} = 20 \text{ MPa} \]

\[ \frac{\sigma_1}{\sigma_{y}} + \frac{\tau_{12}}{\tau_{y}} = \frac{100}{200} + \frac{20}{150} = 0.5 + 0.133 = 0.633 \]

Since the condition is met, yield will occur.

The combined stress test, therefore, allows for the determination of the yield strength at various stress states, which is crucial in the design of components and structures.
A plane-stress yield locus such as Fig. 2-7 may be used to predict yield criteria for an anisotropic material. In anisotropic materials, the yield criterion is not a simple function of stress. The yield stress depends on the orientation of the material axes and the direction of the applied stress. The yield criterion for an anisotropic material is given by the equation:

\[ \sigma = f(\epsilon) \]

where \( \sigma \) is the yield stress and \( \epsilon \) is the strain. The function \( f(\epsilon) \) is different for different directions and orientations of the material.

3.7 Anisotropy in Yielding

The yield criterion for an anisotropic material is more complex than that for a isotropic material. The yield surface is not a simple function of stress, and the yield stress depends on the orientation of the material axes and the direction of the applied stress. The yield criterion for an anisotropic material is given by the equation:

\[ \sigma = f(\epsilon) \]

where \( \sigma \) is the yield stress and \( \epsilon \) is the strain. The function \( f(\epsilon) \) is different for different directions and orientations of the material.

Figure 3-7 shows the yield locus for an anisotropic material.


d) Examination of the paper

1. Display the paper

2. Analyze the paper

3. Formulate the paper

4. Compare the paper

5. Summarize the paper

The yield criterion for an anisotropic material is given by the equation:

\[ \sigma = f(\epsilon) \]

where \( \sigma \) is the yield stress and \( \epsilon \) is the strain. The function \( f(\epsilon) \) is different for different directions and orientations of the material.

Figure 3-7 shows the yield locus for an anisotropic material.
3-8 VYLLD SURFACE AND NORMALTIY

\[\tau = \frac{\tau_0}{Y} \left( \frac{1}{Z} - \frac{1}{Y} \right)\]

Since

\[\frac{(\epsilon_0)}{2} = \frac{\epsilon}{(m_A + 1)}\]

value, the field of the yield stress to the thickness strain
3.10 Invariants of Stress and Strain

The network of equations is quite complex and requires careful analysis. The invariant

\( \tau^2 + \sigma^2 + \pi^2 = 0 \)

represents a critical condition for the stability of the system. The invariant functions

\( \tau, \sigma, \pi \)

are calculated as:

\( \tau = b + c \)

\( \sigma = a + c \)

\( \pi = a + b \)

These invariants are used to simplify the representation of complex states of stress.

3.11 Cayley-Hamilton Theorem

The Cayley-Hamilton theorem is a fundamental result in linear algebra. It states that

\( A^n + \sum_{k=1}^{n-1} \lambda_k A^{n-k} \) = 0

where \( A \) is a square matrix and \( \lambda_k \) are its eigenvalues. This theorem is

\( \lambda \) - \( \lambda \) matrix

and is used to solve systems of linear equations. The theorem is particularly useful in

\( \lambda \) - 2 matrix

the field of numerical analysis and in solving differential equations.
To evaluate $\sigma$ we utilize the effective strain Eq. (3.9), which yields $\sigma = \frac{1}{2} \psi_0$

\[ \psi = \frac{f_0}{f_{ep}} = \frac{f_0}{f_{vp}} = \frac{f_0}{f_{vp}} \]

Thus, the above equations can be written in terms of the actual stress.

The plastic strain increment $\Delta \varepsilon_p$ is obtained from the above equation and the axial deformation $\Delta \varepsilon$ is equal to $\frac{1}{2} \psi_0$

Hence, an incremental basis.

On the basis of total deformation

The condition of continuity of volume in plastic deformation

From which we find

\[ \frac{\varepsilon}{\varepsilon_p} = \frac{\varepsilon_0}{\varepsilon_p} = \frac{\varepsilon_0}{\varepsilon_p} = \frac{\varepsilon_0}{\varepsilon_p} \]

Now necessary to consider the relationship between stress and plastic strain.

#### 3.11 Plastic Stress-Strain Relations

\[ \sigma = 0 \]

Because plastic stress-strain relation is other than

The stresses in the elastic region are given by $\varepsilon = \frac{\sigma}{E}$

\[ \frac{\varepsilon}{\varepsilon_p} = \frac{\varepsilon_0}{\varepsilon_p} = \frac{\varepsilon_0}{\varepsilon_p} \]

For a complete test of $\sigma$, so from Eq. (3.8)
The plastic strain increment is given by the Levy-Kiessling equations, which can be written as:

\[ \varepsilon_p = \frac{\sigma}{\bar{Y}} - 1 \]

where \( \sigma \) is the stress and \( \bar{Y} \) is the yield stress.

The total strain increment is given by the plastic strain increment and the elastic strain increment:

\[ \varepsilon = \varepsilon_p + \varepsilon_{pl} \]

where \( \varepsilon_{pl} \) is the elastic strain increment.

Example: An aluminum thin-walled tube is subjected to a hoop stress of 0.9 MPa. The yield stress of aluminum is 0.4 MPa. From the Levy-Kiessling equations, the plastic strain increment is given by:

\[ \varepsilon_p = \frac{0.9}{0.4} - 1 = 0.25 \]

From the plastic strain increment, the plastic strain is:

\[ \varepsilon_{pl} = \frac{0.9}{0.4} = 2.25 \]

And the total strain is:

\[ \varepsilon = 0.25 + 2.25 = 2.5 \]

Example: A thin-walled pipe is subjected to an internal pressure of 0.4 MPa. The yield stress of the pipe material is 0.8 MPa. From the Levy-Kiessling equations, the plastic strain increment is given by:

\[ \varepsilon_p = \frac{0.4}{0.8} - 1 = 0.5 \]

From the plastic strain increment, the plastic strain is:

\[ \varepsilon_{pl} = \frac{0.4}{0.8} = 0.5 \]

And the total strain is:

\[ \varepsilon = 0.5 + 0.5 = 1 \]
written as

\[ \frac{d\varepsilon_{ij}}{dt} = \frac{3}{2} \frac{d\varepsilon^p_{ij}}{dt} \]  (3-50)

Thus, the stress-strain relations for an elastic-plastic solid are given by

\[ d\varepsilon_{ij} = \frac{1 + \nu}{E} d\sigma_{ij} + \frac{1 + 2\nu}{3} \frac{d\varepsilon^p_{ij}}{dt} \]  (3-51)

Solution of Plasticity Problems

The Levy-Mises and Prandtl-Reuss equations provide relations between the increments of plastic strain and the stresses. The basic problem is to calculate the next increment of plastic strain for a given state of stress when the loads are increased incrementally. If all of the increments of strain are known, then the total plastic strain is simply determined by summation. To do this we have available a set of plastic stress-strain relationships, either Eqs. (3-47) or (3-51), a yield criterion, and a basic relationship for the flow behavior of the material in terms of a curve of \( \bar{\sigma} \) vs. \( \bar{\varepsilon} \). In addition, a complete solution also must satisfy the equations of equilibrium, the strain-displacement relations, and the boundary conditions. The reader is referred to the several excellent texts on plasticity listed at the end of this chapter for examples of detailed solutions.\(^1\) Although the incremental nature of plasticity solutions in the past has resulted in much labor and infrequent application of the available techniques, the current widespread use of digital computers and finite element analysis should make plasticity analysis of engineering problems more commonplace.

3-12 TWO-DIMENSIONAL PLASTIC FLOW—SLIP-LINE FIELD THEORY

In many practical problems, such as rolling and strip drawing, all displacements can be considered to be limited to the xy plane, so that strains in the z direction can be neglected in the analysis. This is known as a condition of plane strain. When a problem is too difficult to an exact three-dimensional solution, a good indication of the stresses often can be obtained by consideration of the analogous plane-strain problem.

Since a plastic material tends to deform in all directions, to develop a plane-strain condition it is necessary to constrain flow in one direction. Constraint can be produced by an external lubricated barrier, such as a die wall (Fig. 3-10a), or it can arise from a situation where only part of the material is deformed and the rigid (elastic) material outside the plastic region prevents the spread of deformation (Fig. 3-10b).

\(^1\) A number of plasticity problems are worked out in great detail in Lubahn and Felgar op. cit., Chaps 8 and 9.

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Figure 3-10 Methods of developing plastic constraint.

If the plane-strain deformation occurs on planes parallel to the xy plane, then

\[ \varepsilon_x = \varepsilon_y = 0 \quad \text{and} \quad \tau_{xz} = \tau_{yz} = 0 \]

Since \( \tau_{xz} = \tau_{yz} = 0 \), it follows that \( \sigma_z \) is a principal stress. From the Levy-Mises equations, Eq. (3-47)

\[ d\varepsilon_z = 0 = \frac{d\varepsilon}{\varepsilon} \left[ \sigma_z - \frac{1}{2} (\sigma_x + \sigma_y) \right] \]

and

\[ \sigma_z = \frac{\sigma_x + \sigma_y}{2} \]  (3-52)

Note that although the strain is zero in the z direction, a restraining stress acts in this direction.

Equation (3-52) could just as well have been written in terms of the principal stresses \( \sigma_1 = (\sigma_1 + \sigma_2)/2 \).

This principal stress will be intermediate between \( \sigma_1 \) and \( \sigma_2 \), so that the maximum-shear-stress yield criterion is given by

\[ \sigma_1 - \sigma_2 = \sigma_0 = 2k \]  (3-53)

where \( k \) is the yield stress in pure shear.

If the value for the intermediate principal stress \( \sigma_1 \) is substituted into the von Mises' yield criterion, Eq. (3-12) it reduces to

\[ \sigma_1 - \sigma_2 = \frac{2}{\sqrt{3}} \sigma_0 \]  (3-54)

However, for the von Mises' yield criterion \( \sigma_0 = \sqrt{3} k \) so that Eq. (3-54) becomes

\[ \sigma_1 - \sigma_2 = 2k \]  (3-55)

Thus, for a state of plane strain the maximum-shear stress and von Mises' yield criteria are equivalent. It can be considered that two-dimensional plastic flow will begin when the shear stress reaches a critical value of \( k \).

Slip-line field theory is based on the fact that any general state of stress in plane strain consists of pure shear plus a hydrostatic pressure. We could show this
and the changes in magnitude of the shear traction to the direction of the slip plane at any point are deduced from the rotation of the slip plane with respect to the shear traction at any point. The slip plane at any point is the plane of maximum shear traction. When the shear traction is in a given plane, the shear traction at any point can be determined if we know the magnitude of the shear traction in that plane and the direction of the plane of maximum shear traction. If the magnitude of the shear traction in any plane can be determined, then the shear traction at any point can be determined.

The slip plane shear traction also allow the determination of stresses.

\[ \gamma - d = \sigma_0 \]
\[ d = \tau_0 \]
\[ \gamma + d = \tau_0 \]

Also, the radius of Mohr's circle is \( r = \frac{d}{d + \sigma} \).

The slip plane shear traction also allow the determination of stresses.

\[ \left( d + \frac{d}{d + \sigma} + \frac{d}{d + \sigma} \right) = \frac{d}{d + \sigma} = \frac{\sigma}{\sigma + \tau_0 + \tau_0} = \frac{\sigma}{\sigma + \tau_0 + \tau_0} = \frac{\sigma}{\sigma + \tau_0 + \tau_0} \]

The slip plane shear traction also allow the determination of stresses.

\[ d - \frac{\sigma}{d + \sigma} - \frac{\sigma}{d + \sigma} = \frac{\sigma}{d + \sigma} \quad \frac{\sigma}{d + \sigma} = \frac{\sigma}{d + \sigma} \]

The slip plane shear traction also allow the determination of stresses.

\[ \begin{bmatrix} d & 0 & 0 \\ 0 & d & \gamma \\ 0 & \gamma & d \end{bmatrix} \]

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\[ \begin{bmatrix} d & 0 & 0 \\ 0 & d & \gamma \\ 0 & \gamma & d \end{bmatrix} \]

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\[ \begin{bmatrix} d & 0 & 0 \\ 0 & d & \gamma \\ 0 & \gamma & d \end{bmatrix} \]
to the use of the theory of plasticity.

The basic equation describing the behavior of the plastic material is the yield condition, which states that the stress in the material is equal to the yield stress of the material. This condition is expressed as:

\[ \sigma = f(\varepsilon) \]

where \( \sigma \) is the stress, \( \varepsilon \) is the strain, and \( f(\varepsilon) \) is the yield function.

The plastic zone is the region where the material has exceeded its yield stress and is undergoing plastic deformation. The size of the plastic zone is determined by the strain gradient and the yield stress of the material.

The plastic flow is described by the plastic strain rate, which is given by:

\[ \dot{\varepsilon} = \frac{\partial f}{\partial \sigma} \]

where \( \dot{\varepsilon} \) is the plastic strain rate, \( \frac{\partial f}{\partial \sigma} \) is the yield function.

The analysis of plasticity problems involves solving the equilibrium equations and the yield condition simultaneously. This is typically done using numerical methods, such as the finite element method (FEM).
This shows that the yield pressure for indentation of a rigid block with a

\[
\sigma = \left( \frac{2}{\mu} + 1 \right) \frac{1}{\mu} \gamma
\]

Since \(\sigma = \phi / V_1\), and the pressure is in tension,

\[
\begin{align*}
\frac{2}{\mu} + 1 &= \gamma - \mu \alpha - \mu \beta - \mu \\
\gamma &= \mu (1 + \alpha) \mu - \mu \\
\end{align*}
\]

The angle from the principal x-axis to the a-line.

From Fig. 3.13, recall that the angle \(\phi\) is measured by the concentric circle

\[
\phi + \sin \theta = \phi \sin \theta + 0 = 0
\]

\[
(1 + \mu) \gamma = 0 = 0
\]

Vertical stress \(\sigma\) is converted to the hydrostatic pressure at the interface into the
discussion (\(d\) is the pressure acting at the block, \(d\) is the pressure acting on the block. \(d\) and the ratio of under the applied load at \(a\) above the same \(a\) is known and that the pressure at \(a\) is the same since the line is straight

\[
1 + a \gamma = 0 = 0
\]

If we write the handy

\[
\gamma = 3 \int \phi d \\
\]

Because \(\gamma\) is straight line, \(d\) is constant from 0 to \(d\).

\[0 = \phi \gamma + d \]

The entire definite equation is as the line from \(d\) to \(\gamma\) as the same. Thus,