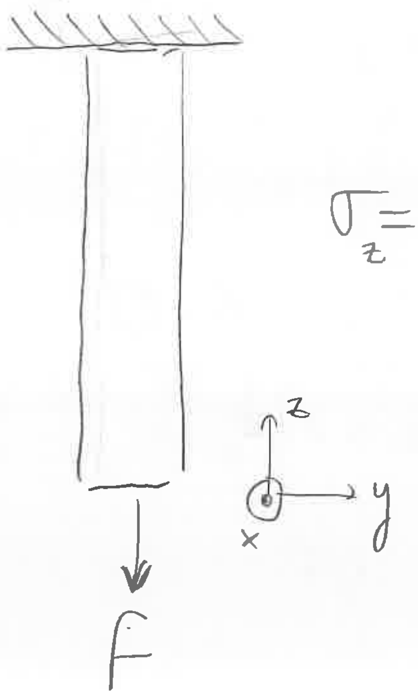
A simple case to analyze



$$\sigma_z = \frac{F}{A}$$

Implications of elasticity equation.

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we have F_z only then we have

$$\frac{\partial \sigma_z}{\partial z} = 0 \rightarrow \sigma_z = \epsilon_z \cdot E = \frac{\partial u_z}{\partial z} \cdot E$$

Thus: $$\frac{\partial}{\partial z}\left(E \cdot \frac{\partial u_z}{\partial z}\right) = 0 \rightarrow E \frac{\partial^2 u_z}{\partial z} = 0$$

What form of a solution for u_z satisfies the above expression? $\rightarrow u_z = Az + B$

If $u_z = Az + B$ → what're A and B?

You find them from the boundary conditions:

$$u_z \Big|_{z=0} = 0 \rightarrow$$ Plug into your trial solution:

$$A \cdot 0 + B = 0 \rightarrow B = 0!$$

Good! Our solution is then: $u_z = Az$

$$u_z = Az$$

$$u_z \Big|_{z=l} = \Delta l \rightarrow \Delta l = A \cdot l \rightarrow \underline{A = \frac{\Delta l}{l}}$$

Strain

Engineering stress-strain vs. true stress-strain

$$\sigma_E = \frac{F}{A_0}$$

F — Instantaneous force

A_0 — Initial cross section

Engineering stress

$$\sigma_T = \frac{F}{A_i}$$

F — Instantaneous force

A_i — Instantaneous cross section

True stress

Engineering strain $\rightarrow$ $\epsilon_E = \dfrac{l_0 - l_F}{l_0}$ Ratio of final extension to the initial length

True strain: $\epsilon_T = \dfrac{\partial l}{l_0} + \dfrac{\partial l}{l_1} + \dfrac{\partial l}{l_2} + \cdots$

$$= \sum_i \underbrace{\frac{\partial l}{l_i}}$$

Ratio of extension to the instantaneous length

Take an instantaneous, infinitesimally small change in length with respect to the instantaneous length:

$$\partial \epsilon_T = \frac{\partial l}{l} \longrightarrow \int_{l_0}^{l_i} d\epsilon_T = \int_{l_0}^{l_i} \frac{\partial l}{l} = \ln \frac{l_i}{l_0}$$

Is it possible to express σ_T in terms of σ_E?

$$\sigma_T = \frac{F}{A_i} = \frac{F}{A_0} \cdot \underbrace{\frac{A_0}{A_1}}_{\propto\, l_i/l_0} = \sigma_E \cdot \left(\frac{l_0 + \partial l}{l_0}\right) = \sigma_E \left(1 + \epsilon_E\right)$$

If one assumes V is cons.

We can also find a relation between ϵ_T and ϵ_E:

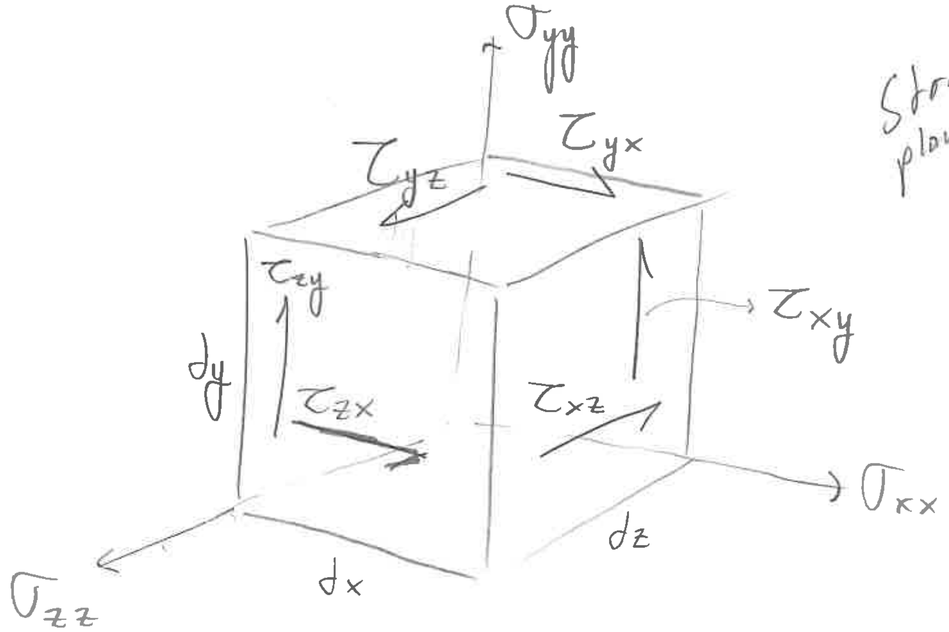
$$\epsilon_E = \frac{l_i - l_0}{l_0} = \frac{l_i}{l_0} - 1 \rightarrow \epsilon_E + 1 = \frac{l_i}{l_0}$$

Take ln of both sides $\longrightarrow$ $\ln(\epsilon_E + 1) = \ln \frac{l_i}{l_0}$

$$\ln(\epsilon_E + 1) = \epsilon_T$$

At any instant, the corresponding ϵ_E and ϵ_T can be expressed in terms of each other.

Components of Stress:



Stress acts on the plane that cuts x

τ_{xy} — Acting Direction

$$\left.\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zx}, \tau_{yz}, \tau_{zy}\right\} \text{Stresses}$$

Matrix representation:

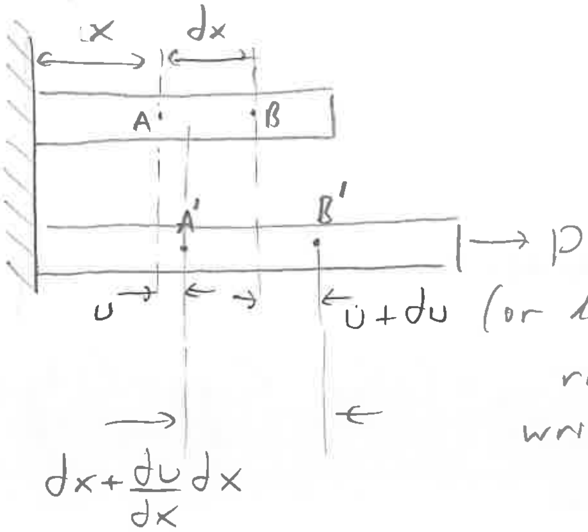
$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \longrightarrow \text{Stress tensor}$$

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \longrightarrow \text{Strain tensor}$$

$$\left.\begin{aligned} \sigma_{ij} &= C_{ijkl}\,\epsilon_{kl} \\ \epsilon_{ij} &= S_{ijkl}\,\sigma_{kl} \end{aligned}\right\} \text{Generalized Hooke's law (Constituent relations)}$$

Definition of strains

Normal strain:



(or let's use chain rule to write $u + \frac{du}{dx}dx$

$$\epsilon_x = \frac{\Delta L}{L} = \frac{A'B' - AB}{AB} = \frac{dx + \frac{du}{dx}dx - dx}{dx} = \frac{\partial u}{\partial x}$$

By some mathematical theorem of tensors, it's proven that:

$$e_{ij} = \underbrace{\frac{1}{2}(e_{ij} + e_{ji})}_{\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} + \frac{1}{2}(\epsilon_{ij} - \epsilon_{ji})$$

Strain tensor

$$\boxed{\text{HW} \rightarrow \text{Show that } \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)}$$ *

Elastic Stress-Strain Relations:

$$\epsilon_x = \frac{\sigma_x}{E} - \left(\frac{\nu\sigma_y}{E} + \frac{\nu\sigma_z}{E}\right)$$

$$= \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right)$$

Similarly $\rightarrow$ $\epsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right)$

$$\epsilon_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right)$$

Shear strains $\rightarrow$ $\tau_{xy} = G\gamma_{xy}$ $\quad \tau_{xz} = G\gamma_{xz}$ $\quad \tau_{yz} = G\gamma_{yz}$

G: Modulus of elasticity in shear

E: Elastic modulus

Isotropic solid $\rightarrow$ Characterized by 3 independent constants E, G, ν

Hydrostatic pressure induces $\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz}$

$\downarrow$

Add the strains to find total volumetric change

$$\underbrace{\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}}_{\Delta:\ \text{Volume Strain}} = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

$$= \underbrace{\frac{1-2\nu}{E}}\, 3\sigma_m$$

$$\frac{3(1-2\nu)}{E} = \frac{1}{K} \rightarrow \overset{\text{Bulk Modulus}}{K} = \frac{E}{3(1-2\nu)}$$

$$\boxed{\text{HW} \rightarrow \text{Show that } G = \frac{E}{2(1+\nu)}} \ *$$

Now, a general representation of an element of a strain tensor exists (for isotropic solid)

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \qquad \left(\text{Using } G = \frac{E}{2(1+\nu)}\right)$$

For example $\rightarrow$ $$\epsilon_{xx} = \frac{1+\nu}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})$$
$$= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})\right]$$

Another example: ϵ_{xy} $(i=x,\ j=y)$

$$\epsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy} - \frac{\nu}{E}\sigma_{kk}$$

There's also σ_{yx} $\rightarrow$ $$\epsilon_{xy} = \underbrace{\left(\frac{1+\nu}{E}\right)}_{\downarrow\ \frac{1}{2G}}\underbrace{(\sigma_{xy}+\sigma_{yx})}_{2\sigma_{xy}}$$

Therefore: $$\epsilon_{xy} = \frac{1}{2G}\cdot(2\sigma_{xy}) = \frac{\sigma_{xy}}{G}$$

Calculation of elastic stresses from strains:

Add the strains and pull stresses to the left:

$$\sigma_x + \sigma_y + \sigma_z = \frac{E}{1-2\nu}(\epsilon_x + \epsilon_y + \epsilon_z)$$

Pull ϵ_x $\rightarrow$ $$\epsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}(\sigma_x+\sigma_y+\sigma_z)$$ This is comp from ϵ_{ij} in tensor notation

Substitute $\sigma_x + \sigma_y + \sigma_z = \frac{E}{(1-2\nu)}(\epsilon_x + \epsilon_y + \epsilon_z)$

you get:

$$\epsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}\left[\frac{E}{(1-2\nu)}(\epsilon_x + \epsilon_y + \epsilon_z)\right]$$

Rearrange for σ_x

$$\sigma_x = \frac{E}{1+\nu}\epsilon_x + \frac{\nu E}{(1+\nu)(1-2\nu)}(\epsilon_x + \epsilon_y + \epsilon_z)$$

In tensor notation:

$$\sigma_{ij} = \frac{E}{1+\nu}\epsilon_{ij} + \underbrace{\frac{\nu E}{(1+\nu)(1-2\nu)}}_{\text{Lame's constant } (\lambda)}\epsilon_{kk}\delta_{ij}$$

Noting that $\epsilon_x + \epsilon_y + \epsilon_z = \Delta$, then

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda\Delta$$

<u>Example (Dieter page 51)</u>

$\epsilon_x = 0{,}004$ and $\epsilon_y = 0{,}001$? What are the principal stresses?

$(E = 200,\ G = 80,\ \nu = 0{,}33)$

Plane stress: $\sigma_3 = 0$

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right]$$
$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right]$$

solve simultaneously to get σ_x

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y)$$
$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x)$$

Now solve the example on Dieter p. 51:

$$\sigma_x = \frac{200}{1-0{,}109}\cdot(0{,}004 + 0{,}33 \cdot 0{,}001) = 965 \text{ MPa} ✓$$

$$\sigma_y = \frac{200}{0{,}891}(0{,}001 + 0{,}0013) = 516 \text{ MPa} ✓$$

Compare it to $\sigma_x = E\epsilon_x = 200 \cdot 0{,}004 = 800 \text{ MPa}$

$\sigma_y = E\epsilon_y = 200 \cdot 0{,}001 = 200 \text{ MPa}$

Incorrect!

One needs to use the tensor notation formula! Or else you compute stresses wrong!

Plane strain: (Watch youtube video for demo)

$$\epsilon_{33} = 0 \rightarrow \epsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right] = 0$$

putting the restriction that $\sigma_3 = \nu(\sigma_1 + \sigma_2)$

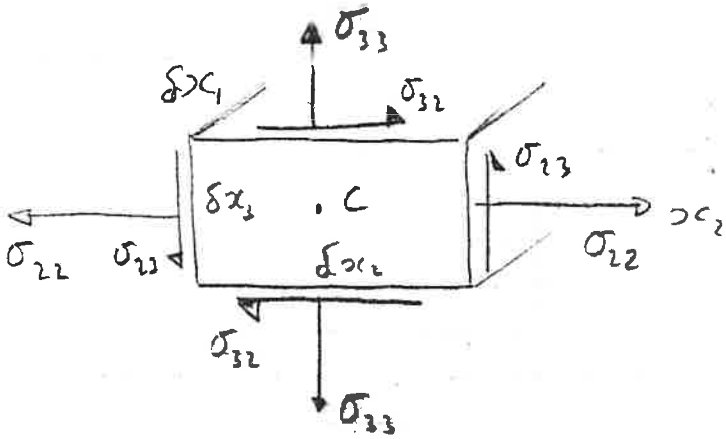
Thus:

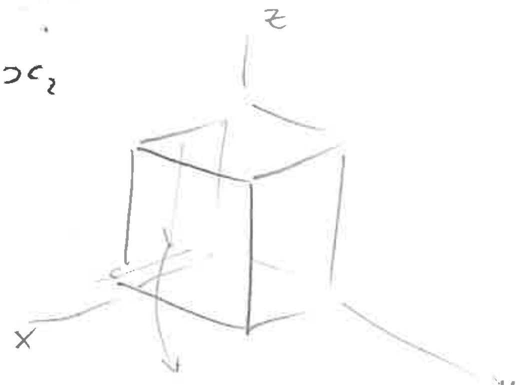
$$\epsilon_x = \frac{1}{E}\left[(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y\right]$$
$$\epsilon_y = \frac{1}{E}\left[(1-\nu^2)\sigma_2 - \nu(1+\nu)\sigma_x\right]$$

$\epsilon_z = 0$

Symmetry of Stress Tensor

Consider moment equilibrium of differential element:





Taking moments about x_1 axis (i.e point C):

$$\sum M_1 = 0: \; 2\left[\sigma_{23}\underbrace{(\delta x_3 \delta x_1)}_{\text{Area of face}}\underbrace{\frac{\delta x_2}{2}}_{\text{Moment arm}}\right] - 2\left[\sigma_{32}(\delta x_2 \delta x_1)\frac{\delta x_3}{2}\right] = 0$$

σ_{yz} — Along z
parallel to surface cutting y

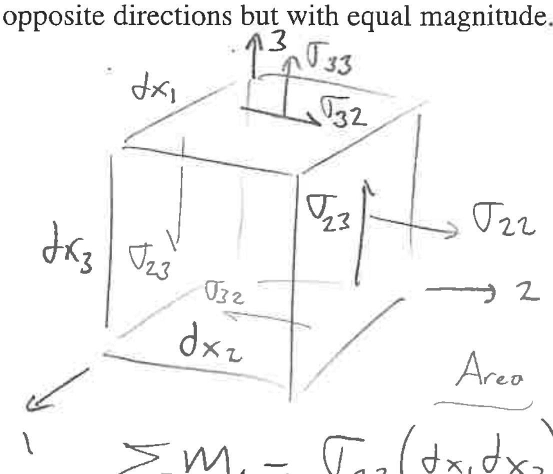
$\Rightarrow \sigma_{23} = \sigma_{32}$

Thus, in general $\quad \sigma_{mn} = \sigma_{nm}$

Stress tensor is symmetric. Six independent components of the stress tensor.

$$\begin{matrix} \sigma_{11} & \sigma_{12} & (=\sigma_{21}) \\ \sigma_{22} & \sigma_{23} & (=\sigma_{32}) \\ \sigma_{33} & \sigma_{31} & (=\sigma_{13}) \end{matrix}$$

Note a positive (tensile) component of stress acts on a face with a positive normal in a positive direction. Thus a stress acting on a negative normal face, in a negative direction is also positive. If the stresses do not vary over the infinitesimal element, σ_{mn} acts on opposite faces, in opposite directions but with equal magnitude.

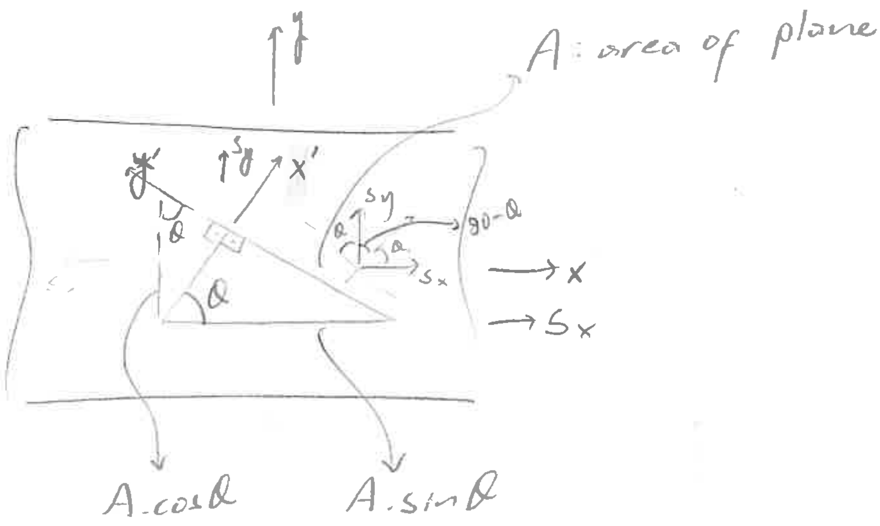


$$\sum M_1 = \sigma_{23}\underbrace{(dx_1 dx_3)}_{\text{Area}}\underbrace{\frac{dx_2}{2}}_{\text{Moment wrt. to center}} - \sigma_{32}(dx_2 dx_1)\frac{dx_3}{2} = 0$$

$$\sigma_{23} = \sigma_{32}$$

3

Stress analysis at a point in 2D



Force balance along x:

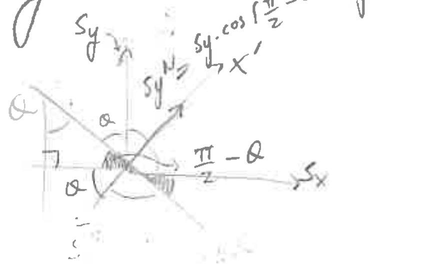
τ_{yx}, does not matter

$$S_x \cdot A = \sigma_x A\cos\theta + \tau_{xy} A \sin\theta$$

Force balance along y:

$$S_y \cdot A = \sigma_y A \sin\theta + \tau_{xy} \cdot A \cdot \cos\theta$$

The normal components of S_x and S_y on the oblique plane are:



$$S_x^N = S_x \cdot \cos\theta \qquad S_y^N = S_y \cdot \sin\theta$$

$$\sigma_x' = S_x \cdot \cos\theta + S_y \cdot \sin\theta$$

Normal to the oblique plane

Insert S_x an S_y into σ_x':

$$\sigma_x' = \sigma_x \cdot \cos^2\theta + \sigma_y \cdot \sin^2\theta + 2\tau_{xy} \sin\theta \cdot \cos\theta$$

The shearing stress is (on the oblique plane)

$$\tau_{xy}' = S_y \underbrace{\cos\theta}_{\cos(\frac{\pi}{2}-\theta)} - S_x \cdot \sin\theta$$

$$\tau_{xy}' = \tau_{xy}(\cos^2\theta - \sin^2\theta) + (\sigma_y - \sigma_x)\sin\theta \cdot \cos\theta$$

If you substitute $\theta + \frac{\pi}{2}$ into σ_x', you get σ_y'

(They are orthogonal!)

If you do it, you get:

$$\sigma_y' = \sigma_x \cos^2\left(\theta + \frac{\pi}{2}\right) + \sigma_y \sin^2\left(\theta + \frac{\pi}{2}\right) + 2\tau_{xy} \sin\left(\theta + \frac{\pi}{2}\right)\cos\left(\theta + \frac{\pi}{2}\right)$$

Since:

$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$ and $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$, we get for

$\sigma_y' \Rightarrow$ $$\sigma_y' = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta$$

Expressing everything in terms of 2θ: (For an elegant organization)

$$\left(\cos^2\theta = \frac{\cos 2\theta + 1}{2} \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2} \right.$$

$$2\sin\theta\cos\theta = \sin 2\theta$$

$$\left. \cos^2\theta - \sin^2\theta = \cos 2\theta \right)$$

$$\left.\begin{aligned} \sigma_x' &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma_y' &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \end{aligned}\right\} \text{Normal stresses on the oblique plane}$$

$$\tau_{xy}' = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \quad \} \text{ Shear stress}$$

H.W: Plot σ_x' vs τ_{xy} and σ_x, σ_y for $\sigma_x = 20$, $\sigma_y = 50$, $\tau_{xy} = 10$

τ_{xy}' vs τ_{xy} and σ_x, σ_y

Stress analysis at a point (cont.'d)

To find θ at which there's no shear:

$(\tau_{x'y'} = 0) \rightarrow$ Set $\tau_{x'y'} = 0 \rightarrow \tau_{xy} \cos 2\theta = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

If you substitute this into σ_x', you get the principal (normal stresses)

To find max. $\tau_{x'y'} \rightarrow \frac{\partial \tau_{x'y'}}{\partial \theta} = 0$

We get: $\tan 2\theta = \frac{\sigma_y - \sigma_x}{2\tau_{xy}}$

Substitute into $\tau_{x'y'}$

and then you find: $\tau_{x'y'}^{max} = \mp \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \right]^{1/2}$

(Do example on page 24, Dieter!)

Arrange σ_x', σ_y' and $\tau_{y'x'}$ ($\tau_{x'y'}$ → it does not matter):

$$\sigma_x' - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{y'x'} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Take squares of both and add them! ; Some algebra here!

$$\left(\sigma_x' - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

It's like: $(x-h)^2 + y^2 = r^2$ } Eqn. of a circle

This is the equation of the Mohr's circle of stress.

If you take the derivative wrt θ and equate the resulting expression to zero, you will find the max shear and principal stress on the oblique plane.

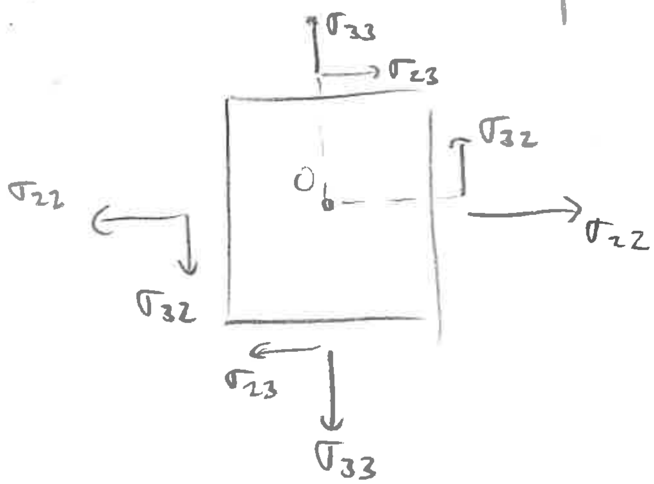
Stress Tensor:

Stress is a second rank tensor.

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Normal components (the rest is shear)

From the balance of moments:



It turns out that the moments should be zero: This requires that

$\sigma_{23} = \sigma_{32}$ as well as

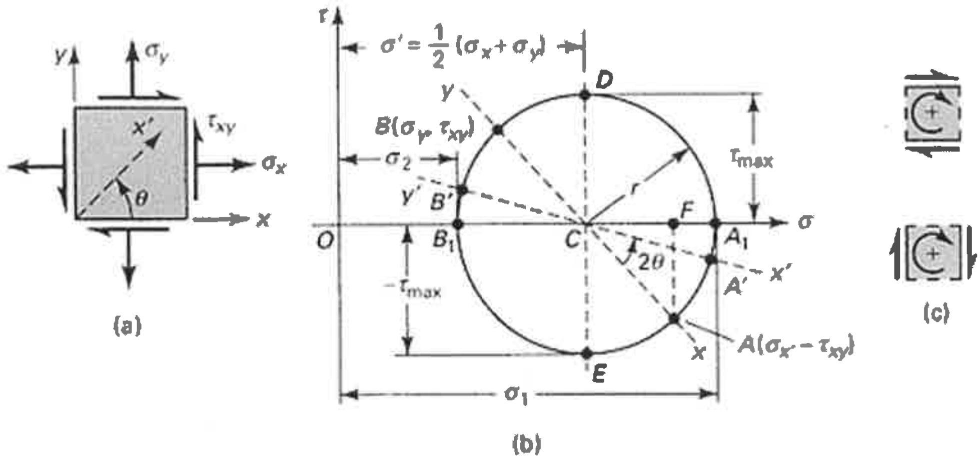
$\sigma_{12} = \sigma_{21}$ and etc.

(Necessary for static equilibrium)

d sin θ

Then we have: $\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix}$

Symmetrical



Given σ_x, σ_y, and τ_{xy} with algebraic sign in accordance with the foregoing sign convention, the procedure for obtaining Mohr's circle (Figure above) is as follows:

1. Establish a rectangular coordinate system, indicating $+\tau$ and $+\sigma$. Both stress scales must be identical.
2. Locate the center C of the circle on the horizontal axis a distance $\frac{1}{2}(\sigma_x + \sigma_y)$ from the origin.
3. Locate point A by coordinates σ_x and $-\tau_{xy}$. These stresses may correspond to any face of an element such as in the Figure. It is usual to specify the stresses on the positive x-face, however.
4. Draw a circle with center at C and of radius equal to CA.
5. Draw line AB through C.

The angles on the circle are measured in the same direction as θ is measured in the Figure above. An angle of 2θ on the circle corresponds to an angle of θ on the element. The state of stress associated with the original x and y planes corresponds to points A and B on the circle, respectively. Points lying on diameters other than AB, such as A' and B', define states of stress with respect to any other set of x' and y' planes rotated relative to the original set through an angle θ.