A

Basic Equations of Elasticity

A.1 STRESS

The state of stress at any point in a loaded body is defined completely in terms of the nine components of stress: $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yx}, \sigma_{yz}, \sigma_{zy}, \sigma_{zx}$, and σ_{xz}, where the first three are the normal components and the latter six are the components of shear stress. The equations of internal equilibrium in terms of the nine components of stress can be derived by considering the equilibrium of moments and forces acting on the elemental volume shown in Fig. A.1. The equilibrium of moments about the x, y, and z axes, assuming that there are no body moments, leads to the relations

$$\sigma_{yx} = \sigma_{xy}, \qquad \sigma_{zy} = \sigma_{yz}, \qquad \sigma_{xz} = \sigma_{zx} \tag{A.1}$$

Equations (A.1) show that the state of stress at any point can be defined completely by the six components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}$, and σ_{zx}.

A.2 STRAIN–DISPLACEMENT RELATIONS

The deformed shape of an elastic body under any given system of loads can be described completely by the three components of displacement u, v, and w parallel to the directions x, y, and z, respectively. In general, each of these components u, v, and w is a function of the coordinates x, y, and z. The strains and rotations induced in the body can be expressed in terms of the displacements u, v, and w. We shall assume the deformations to be small in this work. To derive the expressions for the normal strain components ε_{xx} and ε_{yy} and the shear strain component ε_{xy}, consider a small rectangular element *OACB* whose sides (of lengths dx and dy) lie parallel to the coordinate axes before deformation. When the body undergoes deformation under the action of external load and temperature distribution, the element *OACB* also deforms to the shape $O'A'C'B'$, as shown in Fig. A.2. We can observe that the element *OACB* has two basic types of deformation, one of change in length and the other of angular distortion.

Since the normal strain is defined as change in length divided by original length, the strain components ε_{xx} and ε_{yy} can be found as

$$\varepsilon_{xx} = \frac{\text{change in length of the fiber } OA \text{ which lies in the } x \text{ direction before deformation}}{\text{original length of the fiber}}$$

$$= \frac{\{dx + [u + (\partial u/\partial x)dx] - u\} - dx}{dx} = \frac{\partial u}{\partial x} \tag{A.2}$$

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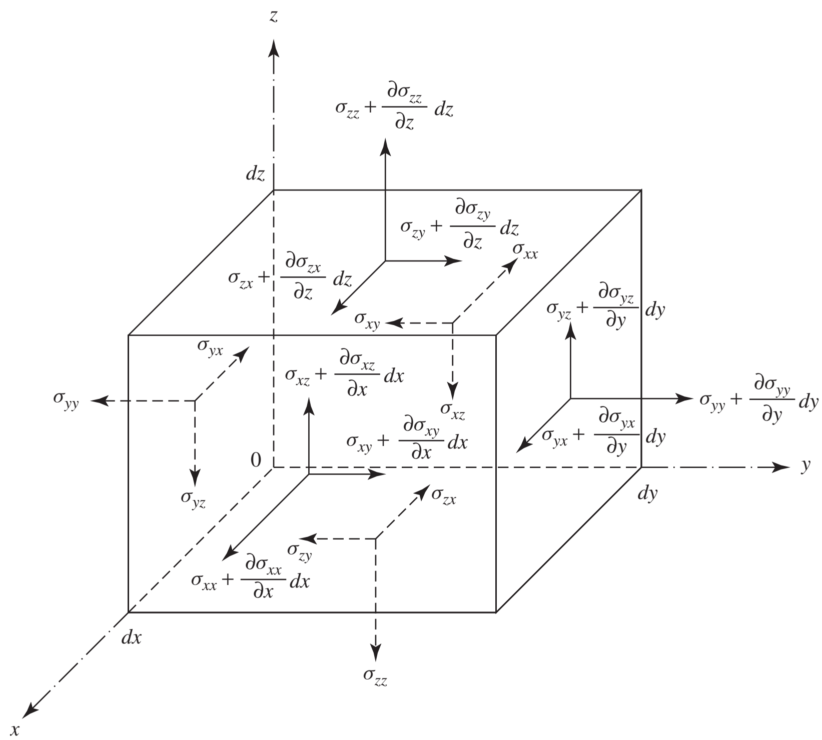

Figure A.1 Stresses on an element of size $dxdydz$.

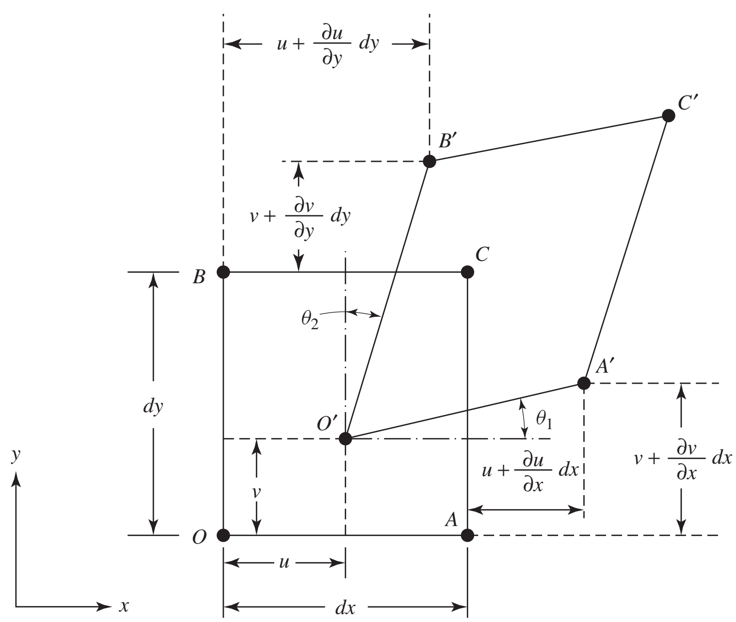

Figure A.2 Deformation of an element.

$$\varepsilon_{yy} = \frac{\text{change in length of the fiber } OB \text{ which lies in the } y \text{ direction before deformation}}{\text{original length of the fiber } OB}$$

$$= \frac{\{dy + [v + (\partial v/\partial y)dy] - v\} - dy}{dy} = \frac{\partial v}{\partial y} \tag{A.3}$$

The shear strain is defined as the decrease in the right angle between fibers *OA* and *OB*, which were at right angles to each other before deformation. Thus, the expression for the shear strain ε_{xy} can be obtained as

$$\varepsilon_{xy} = \theta_1 + \theta_2 \approx \frac{[v + (\partial v/\partial x)dx] - v}{dx + [u + (\partial u/\partial x)dx] - u} + \frac{[u + (\partial u/\partial y)dy] - u}{dy + [v + (\partial v/\partial y)dy] - v} \tag{A.4}$$

If the displacements are assumed to be small, ε_{xy} can be expressed as

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{A.5}$$

The expressions for the remaining normal strain component ε_{zz} and shear strain components ε_{yz} and ε_{zx} can be derived in a similar manner as

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{A.6}$$

$$\varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{A.7}$$

$$\varepsilon_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{A.8}$$

A.3 ROTATIONS

Consider the rotation of a rectangular element of sides dx and dy as a rigid body by a small angle, as shown in Fig. A.3. Noting that $A'D$ and $C'E$ denote the displacements of A and C along the y and $-x$ axes, the rotation angle α can be expressed as

$$\alpha = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{A.9}$$

Of course, the strain in the element will be zero during rigid-body movement. If both rigid-body displacements and deformation or strain occur, the quantity

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{A.10}$$

can be seen to represent the average of angular displacement of dx and the angular displacement of dy, and is called *rotation about the z axis*. Thus, the rotations of an elemental body about the x, y, and z axes can be expressed as

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \tag{A.11}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{A.12}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{A.13}$$

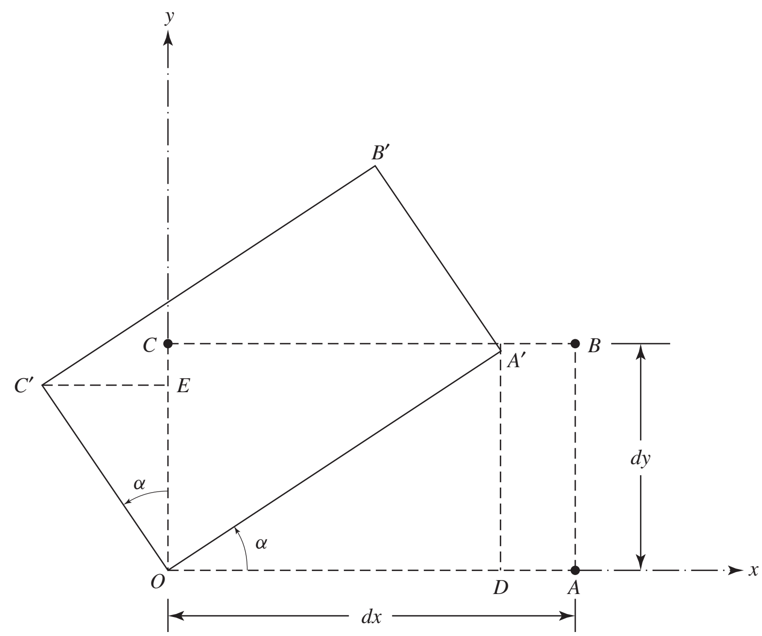

Figure A.3 Rotation of an element.

A.4 STRESS–STRAIN RELATIONS

The *stress–strain relations*, also known as the *constitutive relations*, of an anisotropic elastic material are given by the generalized Hooke's law, based on the experimental observation that strains are linearly related to the applied load within the elastic limit. The six components of stress at any point are related to the six components of strain linearly as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{16} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{26} \\ C_{31} & C_{32} & C_{33} & \cdots & C_{36} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ C_{61} & C_{62} & C_{63} & \cdots & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{Bmatrix} \tag{A.14}$$

where the C_{ij} denote one form of elastic constants of the particular material. Equation (A.14) has 36 elastic constants. However, for real materials, the condition for the elastic energy to be a single-valued function of the strain requires the constants C_{ij} to be symmetric; that is, $C_{ij} = C_{ji}$. Thus, there are only 21 different elastic constants in Eq. (A.14) for an anisotropic material.

For an isotropic material, the elastic constants are invariant, that is, independent of the orientation of the x, y, and z axes. This reduces to two the number of independent elastic constants in Eq. (A.14). The two independent elastic constants, called *Lamé's elastic constants*, are commonly denoted as λ and μ. The Lamè constants are related to C_{ij} as follows:

$$\begin{aligned} C_{11} &= C_{22} = C_{33} = \lambda + 2\mu \\ C_{12} &= C_{21} = C_{31} = C_{13} = C_{32} = C_{23} = \lambda \\ C_{44} &= C_{55} = C_{66} = \mu \\ \text{all other } C_{ij} &= 0 \end{aligned} \tag{A.15}$$

Equation (A.14) can be rewritten for an elastic isotropic material as

$$\begin{aligned} \sigma_{xx} &= \lambda\Delta + 2\mu\varepsilon_{xx} \\ \sigma_{yy} &= \lambda\Delta + 2\mu\varepsilon_{yy} \\ \sigma_{zz} &= \lambda\Delta + 2\mu\varepsilon_{zz} \\ \sigma_{yz} &= \mu\varepsilon_{yz} \\ \sigma_{zx} &= \mu\varepsilon_{zx} \\ \sigma_{xy} &= \mu\varepsilon_{xy} \end{aligned} \tag{A.16}$$

where

$$\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{A.17}$$

denotes the dilatation of the body and denotes the change in the volume per unit volume of the material. Lamé's constants λ and μ are related to Young's modulus E, shear modulus G, bulk modulus K, and Poisson's ratio ν as follows:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{A.18}$$

$$\mathrm{G} = \mu \tag{A.19}$$

$$K = \lambda + \tfrac{2}{3}\mu \tag{A.20}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \tag{A.21}$$

or

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{A.22}$$

$$\mu = \frac{E}{2(1+\nu)} = G \tag{A.23}$$

A.5 EQUATIONS OF MOTION IN TERMS OF STRESSES

Due to the applied loads (which may be dynamic), stresses will develop inside an elastic body. If we consider an element of material inside the body, it must be in dynamic equilibrium due to the internal stresses developed. This leads to the equations of motion of a typical element of the body. The sum of all forces acting on the element shown in Fig. A.1 in the x direction is given by

$$\begin{aligned} \sum F_x &= \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right) dy\,dz - \sigma_{xx}\,dy\,dz + \left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial y} dy\right) dx\,dz - \sigma_{xy}\,dy\,dz \\ &\quad + \left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z} dz\right) dx\,dy - \sigma_{zx}\,dx\,dy \\ &= \frac{\partial \sigma_{xx}}{\partial x} dx\,dy\,dz + \frac{\partial \sigma_{xy}}{\partial y} dx\,dy\,dz + \frac{\partial \sigma_{zx}}{\partial z} dx\,dy\,dz \end{aligned} \tag{A.24}$$

According to Newton's second law of motion, the net force acting in the x direction must be equal to mass times acceleration in the x direction:

$$\sum F_x = \rho \, dx \, dy \, dz \, \frac{\partial^2 u}{\partial t^2} \tag{A.25}$$

where ρ is the density, u is the displacement, and $\partial^2 u/\partial t^2$ is the acceleration parallel to the x axis. Equations (A.24) and (A.25) lead to the equation of motion in the x direction. A similar procedure can be used for the y and z directions. The final equations of motion can be expressed as

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = \rho \, \frac{\partial^2 u}{\partial t^2} \tag{A.26}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \rho \, \frac{\partial^2 v}{\partial t^2} \tag{A.27}$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \, \frac{\partial^2 w}{\partial t^2} \tag{A.28}$$

where u, v, and w denote the components of displacement parallel to the x, y, and z axes, respectively. Note that the equations of motion are independent of the stress–strain relations or the type of material.

A.6 EQUATIONS OF MOTION IN TERMS OF DISPLACEMENTS

Using Eqs. (A.16), the equation of motion, Eq. (A.26), can be expressed as

$$\frac{\partial}{\partial x} (\lambda \, \Delta + 2\mu\varepsilon_{xx}) + \frac{\partial}{\partial y} (\mu \, \varepsilon_{xy}) + \frac{\partial}{\partial z} (\mu\varepsilon_{xz}) = \rho \, \frac{\partial^2 u}{\partial t^2} \tag{A.29}$$

Using the strain–displacement relations given by Eqs. (A.2), (A.4), and (A.8), Eq. (A.29) can be written as

$$\frac{\partial}{\partial x} \left(\lambda \, \Delta + 2 \, \mu \, \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] = \rho \, \frac{\partial^2 u}{\partial t^2} \tag{A.30}$$

which can be rewritten as

$$(\lambda + \mu) \, \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u = \rho \, \frac{\partial^2 u}{\partial t^2} \tag{A.31}$$

where Δ is the dilatation and ∇^2 is the Laplacian operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{A.32}$$

Using a similar procedure, the other two equations of motion, Eqs. (A.27) and (A.28), can be expressed as

$$(\lambda + \mu)\,\frac{\partial \Delta}{\partial y} + \mu \nabla^2 v = \rho\,\frac{\partial^2 v}{\partial t^2} \tag{A.33}$$

$$(\lambda + \mu)\,\frac{\partial \Delta}{\partial z} + \mu \nabla^2 w = \rho\,\frac{\partial^2 w}{\partial t^2} \tag{A.34}$$

The equations of motion, Eqs. (A.31), (A.33), and (A.34), govern the propagation of waves as well as the vibratory motion in elastic bodies.